# THE DETERMINATION OF A LIGHT-TIME ORBIT*

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ABSTRACT

The apparent variation of period arising from the variation of distance of an eclipsing binary from the observer as a result of the gravitational attraction of a distant third component is considered. Woltjer's analytical method is extended and modified. Tables are presented which form the basis of a new graphical method which is both accurate and short.

The following problem was first considered by Woltjer:[1] to determine the elliptic orbit of a star if the distance of the star from an arbitrary fundamental plane (a plane perpendicular to the line of sight) is given by observation as a function of time. This problem arises in the interpretation of the apparent variation of period of an eclipsing binary caused by the changing light-time resulting from the orbital motion of the eclipsing system with reference to a third star; examples are Algol,[2] SV Centauri,[3] and RT Persei.[4] The interpretation of such an observed variation of period may be complicated by the presence of a fourth component, by the rotation of the line of apsides of the small elliptical orbit of the eclipsing pair, or by little-understood changes of period, arising perhaps from changes in the internal structure of one or both of the eclipsing components. The problem of the determination of the light-time orbit will occur with increasing frequency as the observational data become more accurate and extend over greater stretches of time.

If $z$ is the distance of the center of mass of the eclipsing pair from the tangent plane which is perpendicular to the line of sight and which passes through the center of mass of all three stars, then, as is well known,

$$z = r \sin i \sin (v + \omega) . \tag{1}$$

It is more convenient for our purposes to consider $z'$, the distance referred to another (and parallel) plane perpendicular to the line of sight which passes through the *center* of the elliptical orbit of the eclipsing pair about the common center of mass of all three stars. These two reference planes will coincide if $\omega$ is 0° or 180° or if the orbit is circular. If $\tau$ is the light-time and $c$ the velocity of light, then

$$\tau = \frac{z'}{c} = \frac{r \sin i \sin (v + \omega) + a e \sin \omega \sin i}{c} , \tag{2}$$

or

$$\tau = K \frac{1}{\sqrt{(1 - e^2 \cos^2 \omega)}} \left[ \frac{1 - e^2}{1 + e \cos v} \sin (v + \omega) + e \sin \omega \right] , \tag{3}$$

where

$$K = \tfrac{1}{2} (\tau_{\max} - \tau_{\min}) = \frac{a \sin i \sqrt{(1 - e^2 \cos^2 \omega)}}{2.590 \times 10^{10}} , \tag{4}$$

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[1] *B.A.N.*, **1**, 93, 1922.

[2] O. J. Eggen, *Ap. J.*, **108**, 1, 1948.

[3] D. O'Connell, *Riverview College Pub.*, No. 9, 1949; S. Gaposchkin, *Pub. A.S.P.*, **63**, 148, 1951.

[4] R. S. Dugan, *Contr. Princeton U. Obs.*, No. 17, 1938.

$\tau$ and $K$ are in days, and $a$, the semi-major axis, is in kilometers. It can be shown that the expression

$$\frac{1}{\sqrt{(1-e^2\cos^2\omega)}}\left[\frac{1-e^2}{1+e\cos v}\sin(v+\omega)+e\sin\omega\right]$$

varies between $\pm 1$. The quantity $\tau$ is to be taken in the sense of observed time *minus* computed time, where the computed time is $t = t_o + EP'$. If the observational errors are zero and the interpretation correct, then $\tau = \Delta t_{(o-c)}$. The unknowns of the problem, seven in number, are $a \sin i$, $e$, $\omega$, $T$, $P$, $t_o$, and $P'$. The last two elements refer to the

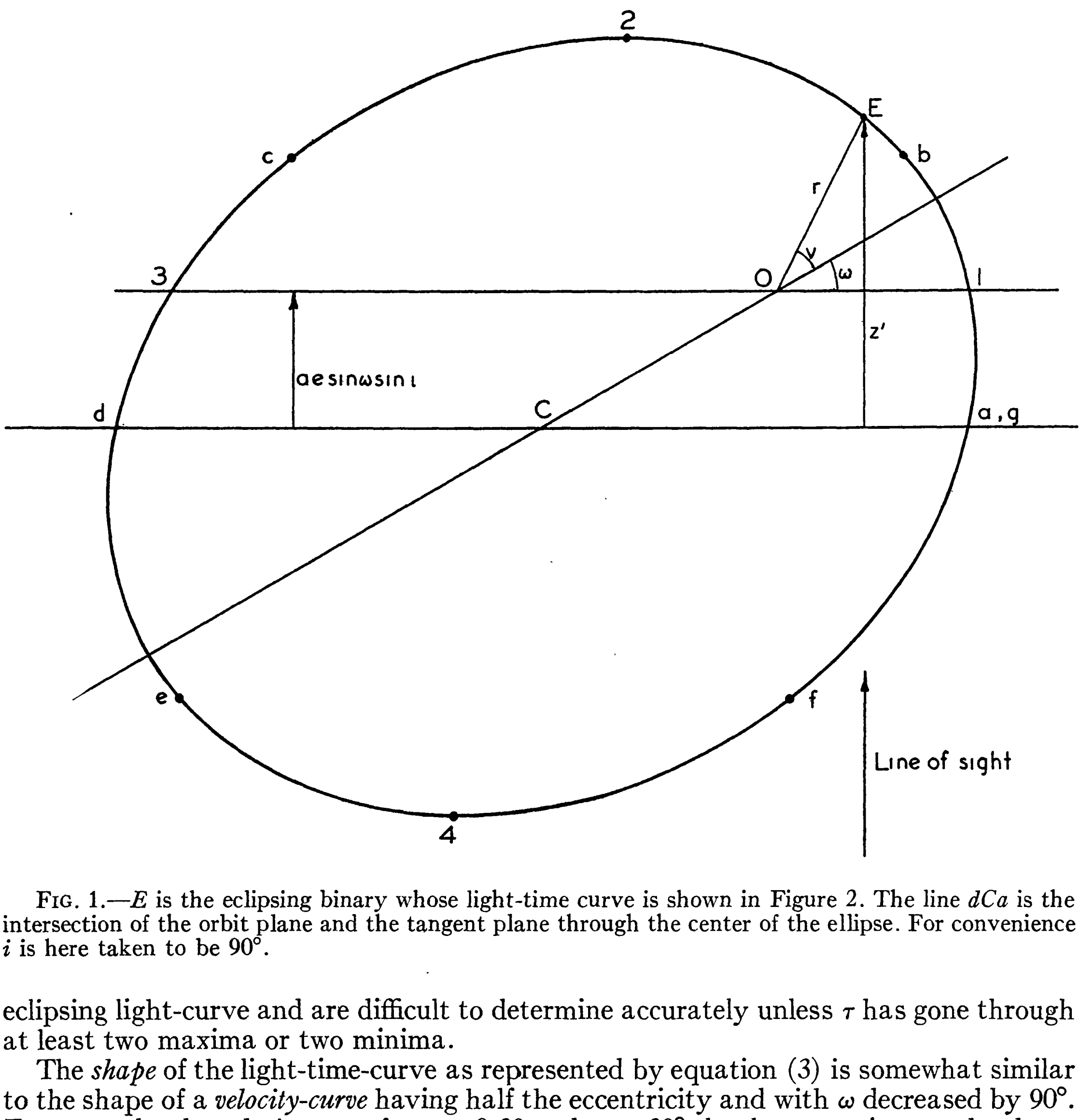


FIG. 1.—$E$ is the eclipsing binary whose light-time curve is shown in Figure 2. The line $dCa$ is the intersection of the orbit plane and the tangent plane through the center of the ellipse. For convenience $i$ is here taken to be 90°.

eclipsing light-curve and are difficult to determine accurately unless $\tau$ has gone through at least two maxima or two minima.

The *shape* of the light-time-curve as represented by equation (3) is somewhat similar to the shape of a *velocity-curve* having half the eccentricity and with $\omega$ decreased by 90°. For example, the *velocity-curve* for $e = 0.30$ and $\omega = 30°$ closely approximates the shape of a light-time-curve with $e = 0.60$ and $\omega = 120°$. The approximation is even better for smaller eccentricities. If a set of standard velocity-curves[5] is available, a very good preliminary value of $e$ and $\omega$ of a light-time-curve can be obtained by fitting the best *velocity-*

[5] E. S. King, *Harvard Ann.*, **81**, 231, 1920.

*curve* to the observations, then doubling the resulting spectroscopic eccentricity, and increasing the spectroscopic $\omega$ by 90°.

If we define $h = (t_d - t_a)/P$, it is easy to show (referring to Fig. 1 and remembering that the areal velocity of the radius vector is constant) that

$$h = \frac{t_d - t_a}{P} = \frac{1}{2} - \frac{e \sin \omega}{\pi \sqrt{(1 - e^2 \cos^2 \omega)}} . \tag{5}$$

Furthermore, if we define $A_1 = (KP)^{-1} \int_{t_a}^{t_d} \tau dt$ and $A_2 = (KP)^{-1} \int_{t_d}^{t_g} \tau dt$, it can be shown,[6] after some considerable algebra, that

$$A_1 - A_2 = \frac{2}{\pi} , \tag{6}$$

and

$$A_1 + A_2 = \frac{-\frac{1}{2} e \sin \omega}{\sqrt{(1 - e^2 \cos^2 \omega)}} , \tag{7}$$

from which

$$h = \frac{1}{2} + \frac{2(A_1 + A_2)}{\pi} , \tag{8}$$

where $A_1$ is positive and $A_2$ negative; they are dimensionless and are shown in Figure 2.

The points of inflection of the light-time-curve occur at points *1* and *3*, where $\tau = Ke \sin \omega / \sqrt{(1 - e^2 \cos^2 \omega)}$ and where the *velocity-curve* shows a maximum or minimum. Following Woltjer,[1] if $\alpha_1$ is the maximum positive slope angle of the light-time-curve and $\alpha_2$ is *minus* the maximum negative slope angle, $\alpha_1$ and $\alpha_2$ both being positive quantities, it follows that

$$e \cos \omega = \frac{\tan \alpha_1 - \tan \alpha_2}{\tan \alpha_1 + \tan \alpha_2} . \tag{9}$$

If $\tau_p$ is the value of $\tau$ at periastron and $T$ is the time of periastron passage, then

$$\tau_p = \frac{K \sin \omega}{\sqrt{(1 - e^2 \cos^2 \omega)}} , \tag{10}$$

and $T$ is the time at that particular value of $\tau$—of the two possibilities—where the curve is steeper. If the orbit is nearly circular, $T$ becomes indeterminate; in such a case, either $t_a$, which is directly obtainable from the light-time-curve, or $t_1$, the time corresponding to the more conventional ascending node, would be preferred; $t_1$ is on the ascending branch of the curve at that point where $\tau = Ke \sin \omega / \sqrt{(1 - e^2 \cos {}^2\omega)}$.

The quantities $\alpha_1$, $\alpha_2$, $K$, and $h$ can be quickly derived from the light-time-curve, so that a solution for the orbital elements is easily made with the aid of equations (9), (5), (4), and (10) in that order. This method may be considered to be a modification of Woltjer's analytical method. Although very quick, it suffers from two disadvantages: (1) it is difficult to determine *accurately* the values of the *maximum* and *minimum* slopes, which means that the value of $e \cos \omega$ as derived from equation (9) may be very uncertain; and (2) there is no way of estimating the errors or uncertainties of the derived orbital elements. For these reasons a numerical-graphical method is given below that overcomes these disadvantages.

We will focus attention at those points on the light-time curve where $\tau = 0.7\,K$, 0, and $-0.7\,K$. These are the points *b* and *c*, *a*, *d*, and *g*, and *e* and *f*, respectively. They are shown in Figures 1 and 2. Values of $(t - T)/P$ can be calculated for these points for any predetermined values of $e$ and $\omega$ by solving equation (3) for $v$, the true anomaly, and

[6] I am indebted to Mr. Robert W. Donselman for pointing out the relationship expressed in eq. (6), which led, in turn, to eq. (7).

then calculating $M$, the mean anomaly, by means of the Allegheny tables,[7] or by means of the well-known relations:

$$\tan \tfrac{1}{2}E = \left(\frac{1-e}{1+e}\right)^{1/2} \tan\tfrac{1}{2}\, v\,, \tag{11}$$

$$\frac{t-T}{P} = \frac{M}{2\pi} = \frac{E - e\sin E}{2\pi}\,. \tag{12}$$

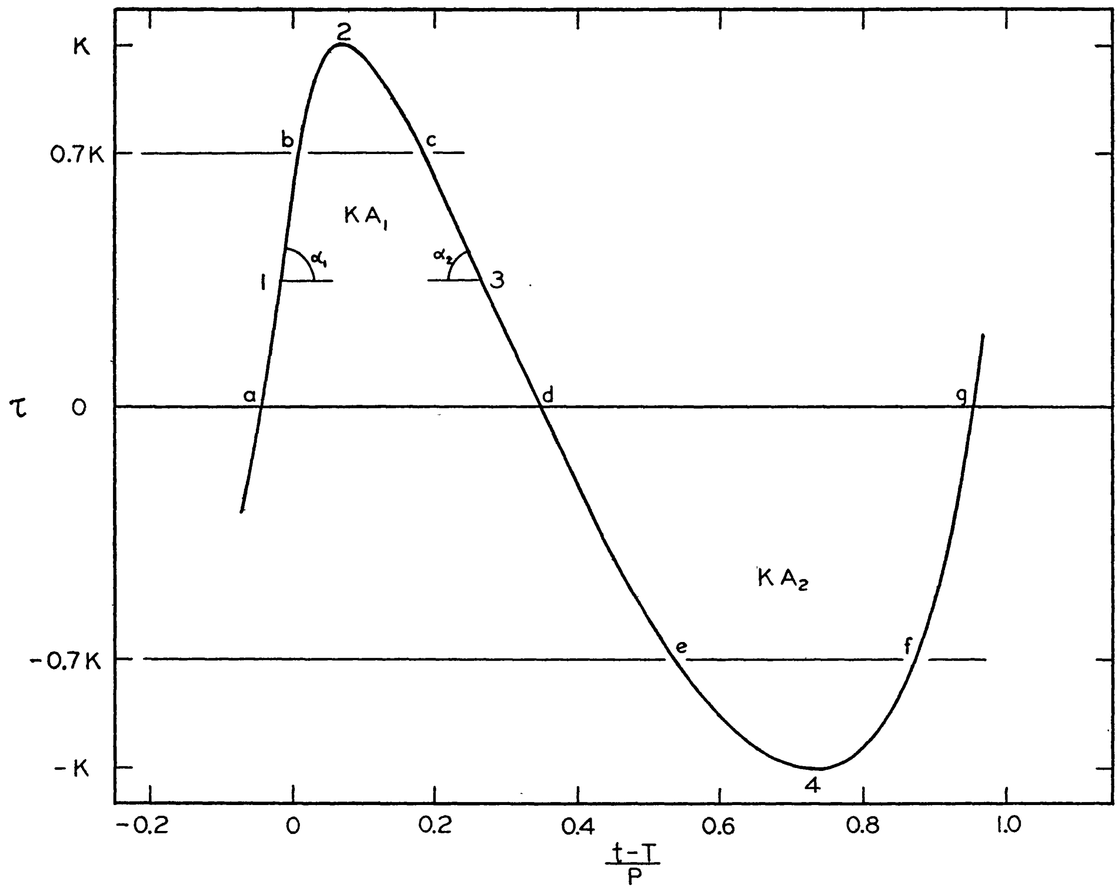


FIG. 2.—The light-time curve for $e = 0.60$ and $\omega = 30°$, corresponding to the orbit shown in Figure 1.

$T$ can be eliminated by taking differences between appropriate pairs of times. The following dimensionless parameters that have been chosen are functions of $e$ and $\omega$ only:

$$h = \frac{t_d - t_a}{P}, \quad s_1 = \frac{t_b - t_a}{P}, \quad p_1 = \frac{t_c - t_b}{P},$$
$$s_2 = \frac{t_d - t_c}{P}, \quad p_2 = \frac{t_f - t_e}{P}, \tag{13}$$
$$s_3 = \frac{t_e - t_d}{P},$$
$$s_4 = \frac{t_g - t_f}{P}.$$

[7] *Pub. Allegheny Obs.*, Vol. **2**, No. 17, 1912.

1952ApJ...116..211I

TABLE 1

$$h=\frac{t_d-t_a}{P}$$

| ω...... | 270 | 255 | 240 | 225 | 210 | 195 | 180 | 165 | 150 | 135 | 120 | 105 | 90 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ω...... | 270 | 285 | 300 | 315 | 330 | 345 | 0 | 15 | 30 | 45 | 60 | 75 | 90 |
| *e* | | | | | | | | | | | | | |
| 0.00... | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 | 0.500 |
| 0.10... | .532 | .531 | .528 | .523 | .516 | .508 | .500 | .492 | .484 | .477 | .472 | .469 | .468 |
| 0.20... | .564 | .562 | .555 | .545 | .532 | .517 | .500 | .483 | .468 | .455 | .445 | .438 | .436 |
| 0.30... | .595 | .593 | .584 | .569 | .549 | .526 | .500 | .474 | .451 | .431 | .416 | .407 | .405 |
| 0.40... | .627 | .624 | .613 | .594 | .568 | .536 | .500 | .464 | .432 | .406 | .387 | .376 | .373 |
| 0.50... | .659 | .655 | .642 | .620 | .588 | .547 | .500 | .453 | .412 | .380 | .358 | .345 | .341 |
| 0.60... | .691 | .687 | .673 | .649 | .612 | .561 | .500 | .439 | .388 | .351 | .327 | .313 | .309 |
| 0.70... | .723 | .719 | .706 | .681 | .640 | .578 | .500 | .422 | .360 | .319 | .294 | .281 | .277 |
| 0.80... | .755 | .751 | .741 | .718 | .677 | .604 | .500 | .396 | .323 | .282 | .259 | .249 | .245 |
| 0.90... | 0.786 | 0.785 | 0.778 | 0.763 | 0.729 | 0.650 | 0.500 | 0.350 | 0.271 | 0.237 | 0.222 | 0.215 | 0.214 |

TABLE 2

$$s_1=\frac{t_b-t_a}{P}; \quad s_2=\frac{t_d-t_c}{P}; \quad s_3=\frac{t_e-t_d}{P}; \quad s_4=\frac{t_g-t_f}{P}$$

| ω for $s_1$...... | 0 | 15 | 30 | 45 | 60 | 75 | 90 | 105 | 120 | 135 | 150 | 165 | 180 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ω for $s_2$...... | 180 | 165 | 150 | 135 | 120 | 105 | 90 | 75 | 60 | 45 | 30 | 15 | 0 |
| ω for $s_3$...... | 180 | 195 | 210 | 225 | 240 | 255 | 270 | 285 | 300 | 315 | 330 | 345 | 360 |
| ω for $s_4$...... | 360 | 345 | 330 | 315 | 300 | 285 | 270 | 255 | 240 | 225 | 210 | 195 | 180 |
| *e* | | | | | | | | | | | | | |
| 0.00........ | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 |
| .10........ | .112 | .111 | .111 | .112 | .114 | .116 | .119 | .122 | .125 | .128 | .131 | .133 | .135 |
| .20........ | .101 | 100 | 100 | 101 | 105 | .109 | .114 | .120 | .126 | .133 | .138 | .142 | .146 |
| .30........ | .090 | .088 | .088 | .090 | .095 | .102 | .110 | .118 | .128 | .137 | .145 | .152 | .157 |
| .40........ | .079 | .076 | .076 | .080 | .086 | .095 | .105 | .116 | .128 | .140 | .151 | .161 | .168 |
| .50........ | .068 | .063 | .064 | .070 | .078 | .089 | .101 | .114 | .128 | .143 | .157 | .170 | .179 |
| .60........ | .057 | .051 | .053 | .060 | .071 | .083 | .096 | .111 | .127 | .144 | .162 | .178 | .190 |
| .70........ | .045 | .039 | .043 | .052 | .064 | .077 | .092 | .107 | .124 | .143 | .164 | .185 | .201 |
| .80........ | .034 | .027 | .034 | .046 | .060 | .073 | .087 | .102 | .118 | .138 | .162 | .190 | .213 |
| 0.90........ | 0.023 | 0.017 | 0.030 | 0.046 | 0.059 | 0.071 | 0.082 | 0.094 | 0.108 | 0.126 | 0.151 | 0.187 | 0.224 |

| ω for $s_1$...... | 180 | 195 | 210 | 225 | 240 | 255 | 270 | 285 | 300 | 315 | 330 | 345 | 360 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ω for $s_2$...... | 360 | 345 | 330 | 315 | 300 | 285 | 270 | 255 | 240 | 225 | 210 | 195 | 180 |
| ω for $s_3$...... | 0 | 15 | 30 | 45 | 60 | 75 | 90 | 105 | 120 | 135 | 150 | 165 | 180 |
| ω for $s_4$...... | 180 | 165 | 150 | 135 | 120 | 105 | 90 | 75 | 60 | 45 | 30 | 15 | 0 |
| *e* | | | | | | | | | | | | | |
| 0.00........ | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 | 0.123 |
| .10........ | .135 | .135 | .135 | .134 | .133 | .131 | .128 | .125 | .122 | .119 | .116 | .114 | .112 |
| .20........ | .146 | .147 | .147 | .146 | .142 | .138 | .133 | .127 | .120 | .114 | .109 | .104 | .101 |
| .30........ | .157 | .159 | .159 | .156 | .151 | .145 | .137 | .128 | .119 | .110 | .102 | .095 | .090 |
| .40........ | .168 | .171 | .171 | .167 | .160 | .152 | .142 | .130 | .119 | .107 | .095 | .086 | .079 |
| .50........ | .179 | .183 | .182 | .177 | .169 | .158 | .146 | .133 | .119 | .104 | .090 | .077 | .068 |
| .60........ | .190 | .195 | .194 | .186 | .176 | .164 | .151 | .136 | .120 | .103 | .085 | .069 | .057 |
| .70........ | .201 | .208 | .204 | .195 | .183 | .169 | .155 | .140 | .123 | .104 | .083 | .062 | .045 |
| .80........ | .213 | .220 | .213 | .200 | .187 | .173 | .160 | .145 | .129 | .109 | .084 | .057 | .034 |
| 0.90........ | 0.224 | 0.230 | 0.217 | 0.201 | 0.188 | 0.176 | 0.164 | 0.152 | 0.139 | 0.121 | 0.096 | 0.059 | 0.023 |

These seven parameters are tabulated in Tables 1, 2, and 3 as functions of $e$ and $\omega$. The accuracy of the tables is better than 0.0006.

INSTRUCTIONS TO THE COMPUTER

Given the observations, draw the $\tau$ ($= \Delta t_{[o-c]}$)—or light-time-curve—with assumed values of $t_o$ and $P'$ (the period of eclipses). Both the position and (essentially) the slope of the $\tau = 0$ axis are dependent upon the assumed values of $t_o$ and $P'$, and the correct choice of these elements may be quite difficult to make, unless there are at least two well-defined maxima or minima. Equations (6) and (8) may be useful at this point in fixing these elements. $t_o$ is so chosen that the $\tau = 0$ axis is the mean axis, halfway between $\tau_{max}$ and $\tau_{min}$. The light-time orbit period, $P$, can now be read from the curve; the times corresponding to the points $a$, $b$, $c$, $d$, $e$, $f$, and $g$ can also be read from the curve and the seven parameters calculated as defined by equations (13).

The value of each parameter, in turn, may now be expressed as a curve on the $\omega$–$e$ plane by interpolating in the appropriate table. One (or more) of these parameters may be omitted if the observational data necessary to its determination are weak or missing. If the fundamental assumptions are correct and the observations without error, the curves so plotted should meet at a point, thereby fixing the values of $e$ and $\omega$. In any practical case the curves will not meet at a point, and some kind of mean intersection must be adopted. An example is the solution for the 188.4-year light-time curve of Algol as shown in Figure 3. New values of the seven parameters should be interpolated from the tables by making use of the adopted values of $e$ and $\omega$; and, by adopting the time at some well-determined point on the curve, it will be found possible to represent the seven points and compare them with the observations. A horizontal shift (or correction) in the time scale may be conveniently made here. A correction to either or both the maximum and the minimum values of $\tau$, and hence $K$ and $t_o$, may also be made. If the agreement with the observations is not entirely satisfactory, the tables will indicate what changes in $e$ and $\omega$ are necessary and possible. The value of $T$, the time of periastron passage, may be obtained by making use of equation (10); the computer should be familiar with the remarks immediately following this equation. If $T$ occurs very close to the time of $\tau_{max}$ or $\tau_{min}$ (points *2* or *4*), a more satisfactory determination of $T$ is possible by computing $v_2$ or $v_4$, the true anomalies at these points, from[8]

$$\sin(v_2 + \omega) = \sqrt{(1 - e^2 \cos^2 \omega)}\,; \qquad (14)$$

$$\sin(v_4 + \omega) = -\sqrt{(1 - e^2 \cos^2 \omega)}\,. \qquad (15)$$

The mean anomaly and $t_2 - T$ (or $t_4 - T$) follow immediately from equations (11) and (12) or by making use of the Allegheny tables.[7] The value of $a \sin i$ is calculated from equation (4).

A representation of the final adopted orbital elements may now be made by making use of equation (3). The writer has found it convenient and practical to calculate $\tau$ for values of $v$ every 15° from 0° to 345° inclusive. The times for these true anomalies can be quickly obtained with the aid of the Allegheny tables. If the computer at this point feels that he must indulge in a least-squares correction, analytical expressions for the differential coefficients have been published by Scott.[9] In view of the difficulty in accurately fixing $t_o$ and $P'$ and because of complications that may arise from other phenomena that would cause a variation of period, such a least-squares solution would not, in general, be required.

[8] As is well known from spectroscopic binary theory, $\cos(v + \omega) = -e \cos \omega$ at these points, which correspond to the times when the velocity-curve crosses the $\gamma$ axis.

[9] *Harvard Bull.*, No. 912, 1940. I am informed by Dr. O. J. Eggen that a correction to one of Scott's formulae has been published by Vasilieva, *Astr. Circ. Acad. Sci. USSR*, No. 75, 1948. This reference is not immediately available to me.

The writer's original interest in this problem was stimulated by Dugan and Wright's paper[10] concerning the remarkable variation of period of the 1.66-day eclipsing binary SV Centauri. Their published times of minima can be satisfied with tolerable accuracy by a light-time orbit with $e = 0.50$ and $\omega = 90°$. Unfortunately, the resulting mass function is 6800; it would be unrealistic to attach any reality to the assumptions leading to such a result. O'Connell[3] has suggested a mass function of the order of at least 10,000

TABLE 3

$$p_1 = \frac{t_c - t_b}{P}; \qquad p_2 = \frac{t_f - t_e}{P}$$

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\omega$ for $p_1$ | 270 | 255 | 240 | 225 | 210 | 195 | 180 | 165 | 150 | 135 | 120 | 105 | 90 |
| $\omega$ for $p_1$ | 270 | 285 | 300 | 315 | 330 | 345 | 0 | 15 | 30 | 45 | 60 | 75 | 90 |
| $\omega$ for $p_2$ | 90 | 75 | 60 | 45 | 30 | 15 | 0 | 345 | 330 | 315 | 300 | 285 | 270 |
| $\omega$ for $p_2$ | 90 | 105 | 120 | 135 | 150 | 165 | 180 | 195 | 210 | 225 | 240 | 255 | 270 |
| $e$ | | | | | | | | | | | | | |
| 0.00 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 | 0.253 |
| .10 | .276 | .275 | .273 | .269 | .265 | .259 | .253 | .247 | .242 | .237 | .233 | .231 | .230 |
| .20 | .299 | .297 | .293 | .286 | .276 | .265 | .253 | .241 | .230 | .221 | .214 | .209 | .208 |
| .30 | .321 | .319 | .313 | .303 | .288 | .272 | .253 | .235 | .218 | .204 | .193 | .187 | .185 |
| .40 | .344 | .341 | .334 | .320 | .302 | .279 | .253 | .228 | .205 | .186 | .173 | .165 | .162 |
| .50 | .367 | .364 | .355 | .339 | .316 | .287 | .253 | .220 | .190 | .167 | .152 | .142 | .140 |
| .60 | .390 | .387 | .377 | .360 | .333 | .296 | .253 | .210 | .173 | .147 | .129 | .120 | .117 |
| .70 | .412 | .409 | .400 | .383 | .353 | .309 | .253 | .197 | .153 | .124 | .106 | .097 | .094 |
| .80 | .435 | .433 | .425 | .409 | .379 | .327 | .253 | .179 | .127 | .097 | .081 | .074 | .071 |
| 0.90 | 0.458 | 0.456 | 0.452 | 0.441 | 0.416 | 0.360 | 0.253 | 0.146 | 0.090 | 0.066 | 0.055 | 0.050 | 0.049 |

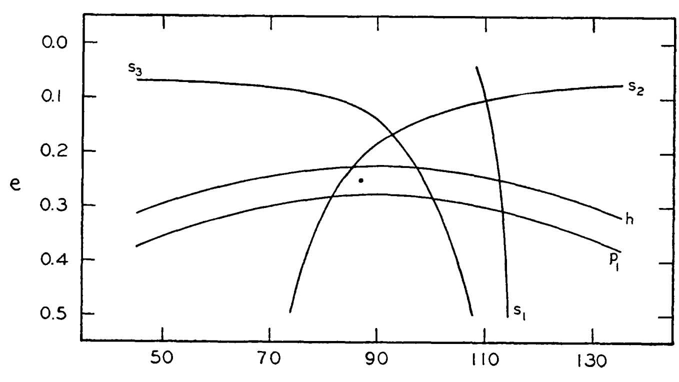


FIG. 3.—The solution for the 188.4-year variation of period of Algol. Eggen's solution is the filled circle. The uncertainties in $e$ and $\omega$ are greater than indicated because of the uncertainties of both $P$ and $P'$.

and has deduced a mass function of 49 for still another hypothetical component by considering the systematic run of the residuals from an empirical parabolic formula. This latter result would also seem to be an unsatisfactory answer. The cause of the variation of period of this system, which has, by far, the greatest known amplitude of variation, must still be considered to be unknown. Until the correct explanation is forthcoming a certain amount of doubt must be attached to every other light-time orbit solution.

[10] *Contr. Princeton U. Obs.*, No. 19, 1939.